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ON THE DIOPHANTINE EQUATION $32^x + 49^y = z^2$

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Abstract

In this paper, we show that the Diophantine equation $32^x + 49^y = z^2$ has a unique non-negative integer solution. The solution (x, y, z) is (1, 1, 9). This results implies that (5, 2, 9) is a solution (x, y, z) of the Diophantine equation $2^x + 7^y = z^2$, where x, y, and z are non-negative integer.

1. Introduction

In 2007, Acu [1] showed that the Diophantine equation $2^x + 5^y = z^2$ has only two solutions in non-negative integers. The solutions (x, y, z) are (3, 0, 3) and (2, 1, 3). In 2011, Suvarnamani et al. [6] showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. Moreover, in the same year, Suvarnamani [5] found some non-negative integer solutions of the Diophantine

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equation of type $2^x + p^y = z^2$, where p is a positive prime number. In 2012, Chotchaisthit [3] found all non-negative integer solutions of the Diophantine equation of type $4^x + p^y = z^2$, where p is a positive prime number. In this paper, we will show that (1, 1, 9) is a unique solution (x, y, z) of the Diophantine equation $32^x + 49^y = z^2$, where x, y, and z are non-negative integers.

2. Preliminaries

In 1844, Catalan [2] conjectures that the Diophantine equation $a^x + b^y = 1$ has a unique integer solution with $\min\{a, b, x, y\} > 1$. The solution (a, b, x, y) is (3, 2, 2, 3). This conjecture was proven by Mihăilescu [4] in 2004.

Proposition 2.1 ([4])**.** (3, 2, 2, 3) *is a unique solution* (a, b, x, y) *of the Diophantine equation* $a^x + b^y = 1$, *where* a, b, x, *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Proposition 2.2 ([5])**.** (3, 0, 3) *is a solution* (x, y, z) *of the Diophantine equation* $2^x + 7^y = z^2$, *where* x, y, *and* z *are non-negative integer*.

3. Results

In this section, we prove that the Diophantine equation $32^x + 49^y = z^2$ has a unique non-negative integer solution. The solution (x, y, z) is (1, 1, 9). This results implies that (5, 2, 9) is a solution (x, y, z) of the Diophantine equation $2^x + 7^y = z^2$, where x, y, and z are non-negative integer.

Theorem 3.1. (1, 1, 9) *is a unique solution* (x, y, z) *of the Diophantine equation* $32^x + 49^y = z^2$, *where* x, y, *and* z *are non-negative integers*.

Proof. We will divide the number x into two cases.

Case $x = 0$**.** We focus on the equation $1 + 49^y = z^2$. Then $(z-1)(z+1) = 7^{2y}$. Thus, $z - 1 = 7^u$, where u is a non-negative integer. Then $z + 1 = 7^{2y-u}$. It follows that $7^{2y-u} - 7^u = 2$. Then $7^u(7^{2y-2u} - 1) = 2$. This implies that $u = 0$. Then $7^{2y} - 1 = 2$. Thus, $7^{2y} = 3$. This is impossible.

Case $x \geq 1$**.** In this case, z is odd. We consider the equation $32^x + 49^y = z^2$ as the equation $2^{5x} + 7^{2y} = z^2$. Then $(z - 7^y)(z + 7^y) = 2^{5x}$. Then $z - 7^y = 2^w$, where w is a non-negative integer. Note that 7^y is odd. We have $w \neq 0$. Moreover, $z + 7^y = 2^{5x-w}$. It follows that $2^{5x-w} - 2^w = 2(7^y)$. Then $2^w(2^{5x-2w} - 1) = 2(7^y)$. Then $w = 1$. It follows that $2^{5x-2} - 1 = 7^y$. If $y = 0$, then $x = 0.6$. Thus, $y \geq 1$. By Proposition 2.1, we obtain that $x = 1$ or $y = 1$. Now, we note that $x = 1$ if and only if $y = 1$. Thus, $x = 1$ and $y = 1$. Then $z^2 = 32 + 49 = 81$. Hence, $z = 9$.

Therefore, $(x, y, z) = (1, 1, 9)$.

Corollary 3.2. $(5, 2, 9)$ *is a solution* (x, y, z) *of the Diophantine equation* $2^x + 7^y = z^2$, *where* x, y, *and* z *are non-negative integer*.

Proof. By Theorem 3.1, we obtain that $32^1 + 49^1 = 9^2$. This implies that $2^5 + 7^2 = 9^2$. Therefore, $(5, 2, 9)$ is a solution (x, y, z) of the Diophantine equation $2^x + 7^y = z^2$, where x, y, and z are non-negative integer.

4. Open Problem

By Proposition 2.2 and Corollary 3.2, we know that (3, 0, 3) and (5, 2, 9) are two solutions (x, y, z) of the Diophantine equation $2^x + 7^y = z^2$, where x, y, and z are non-negative integer. However, we may question that "what is the set of all solutions (x, y, z) of the Diophantine equation $2^x + 7^y = z^2$, where x, y, and z are non-negative integer?".

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